IMSC 2058 Solution for Homework 7

Ex 5.12

Let $\{x_n\}$ be Cauchy in X. there exists N such that $d(x_m, x_n) < \varepsilon/2$ for all $m, n \geq N$.

Since X is compact metric, it is sequentially compact: Every sequence in X has a subsequence converging to some point in X. Thus, there exists K such that $d(x_{n_k}, x) < \varepsilon/2$ for all $k \geq K$.

Choose $k \geq K$ with $n_k \geq N$. Now, for any $n \geq n_k$, we have $n \geq N$ and $n_k \geq N$, so $d(x_n, x_{n_k}) < \varepsilon/2$. Then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore the Cauchy sequence $\{x_n\}$ converges in X.

Ex 5.17

Assume X is compact. Let $\{F_\alpha\}_{\alpha \in I}$ be any family of closed subsets of X with the finite intersection property. Then $U_\alpha = X \setminus F_\alpha$ is open in X for $\alpha \in I$.

Suppose $\bigcap_{\alpha \in I} F_\alpha = \emptyset$. Then $\bigcup_{\alpha \in I} U_\alpha = X$, so $\{U_\alpha\}$ is an open cover of X. Then there is a finite subcover: $\bigcup_{k=1}^n U_{\alpha_k} = X$ for some finite $\{\alpha_1, \ldots, \alpha_n\}$. This implies $\bigcap_{k=1}^n F_{\alpha_k} = X \setminus \bigcup_{k=1}^n U_{\alpha_k} = \emptyset$. But this contradicts the finite intersection property. Hence $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

Assume every family of closed subsets with the finite intersection property has non-empty intersection. Then we will show every open cover of X has a finite subcover.

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of X. Define the closed sets $F_\alpha = X \setminus U_\alpha$ for each $\alpha \in A$. Suppose that $\{U_\alpha\}_{\alpha \in A}$ has no finite subcover. Then, for every finite subcollection, $\bigcup_{k=1}^n U_{\alpha_k} \neq X$, so $\bigcap_{k=1}^n F_{\alpha_k} \neq \emptyset$. Thus, $\{F_\alpha\}_{\alpha \in A}$ has the finite intersection property. However

$$\bigcap_{\alpha \in A} F_\alpha = X \setminus \bigcup_{\alpha \in A} U_\alpha = X \setminus X = \emptyset$$

which contradict to $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$. Therefore, the open cover must have a finite subcover, so X is compact.